4. de Groot, S. R., Thermodynamics of Irreversible Processes. Gostekhizdat, Moscow, 1956.

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CHARACTER OF THE LOSS OF STABILITY IN A NONEQUILIBRIUM MAGNETIZED PLASMA

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The character of the loss of stability in a magnetized nonequilibrium plasma in a bounded region is examined. The influence of thermal conductivity and nonlinear effects are taken into account. It is shown that both magnetically soft and magnetically hard modes of the loss of stability can take place. With small parameter values of the supercritical state the rise of self-oscillations is possible. The critical value of the Hall parameter corresponding to the beginning of ionization instability [1, 2] and its dependence on the boundaries, were examined using the linear approximation in [3-5]. The spectrum for the linear problem was examined neglecting thermal conductivity [3] and taking it into account [5]. It was established that the critical value of the Hall parameter is identical in the presence of boundaries and in the case of an infinite medium. Numerical computations of the ionization instability process are given in [6, 7). The bibliography of early works on ionization instability can be found in the review [8].

1. Let us consider the behavior of a nonequilibrium magnetized plasma in a bounded region. An infinite channel extends in the direction of the y and z axes and it is bounded in the direction of the x axis by nonconducting walls, separated by a distance b. A constant magnetic field B is directed along the z axis. It is assumed that ionization equilibrium exists anf the effect of the induced magnetic field is neglected. If the electron temperature T considerably exceeds the heavy particle temperature T_a, then the state of the plasma is defined by the following system of equations:

$$\operatorname{rot}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{j} = 0 \tag{1.1}$$

$$I\left(\frac{\partial n}{\partial t} + \mathbf{U}\,\nabla n\right) + \frac{3}{2}nk\left(\frac{\partial T}{\partial t} + \mathbf{U}\,\nabla T\right) + \operatorname{div}\mathbf{q} = \frac{j^2}{\sigma} - \frac{3}{2}kT\delta n\nu$$

$$U = -\frac{j}{en}, \quad \mathbf{j} + \mathbf{j}\times\boldsymbol{\Omega} = \sigma(n, T)\,\mathbf{E}, \quad \mathbf{q} + \mathbf{q}\times\boldsymbol{\Omega} = -\lambda(n, T)\,\nabla T$$

Here I is the ionization potential, U is the directional electron velocity, σ and λ are the coefficients of electric and thermal conductivity, respectively, j is the electric current density, ν is the frequency of collisions between the electrons and heavy particles, δ is the portion of energy transferred at the collision with a heavy particle, E is the electric field strength, $\Omega = \omega/\nu\,(n,\ T)$ is the Hall parameter, ω is the electron cyclotron frequency.

Since the presence of an ionization equilibrium is assumed, the concentration of electrons n depends on their temperature as is given in the Saha equation. In the absence of ionization equilibrium, the system of equations (1.1) must be completed by the equation of ionization kinetics.

We introduce the potential Φ

$$j_x = \frac{\partial\Phi}{\partial y}, \quad j_y = j_0 - \frac{\partial\Phi}{\partial x}, \quad j_0 = \text{const}$$

As a typical electron temperature T_0 for a given value of j_0, we take the temperature obtained by solving the equation

$$^3/_2 \delta k T_0 n_0 (T_0)\, \nu_0 (n_0, T_0) = j_0^2 / \sigma_0 (n_0, T_0)$$

We denote the values of all parameters of the medium at a given temperature by a zero subscript $(n(T_0) \equiv n_0)$. We choose

$$l_x = b, \quad \tau_x = \frac{I n_0 \sigma_0}{j_0^2}, \quad U_x = \frac{b}{\tau_x}$$

as characteristic values and introduce the dimensionless parameters

$$T^+ = \frac{T - T_0}{T_0}, \quad \theta = \frac{n - n_0}{n_0}, \quad \Phi^+ = \frac{\Phi}{j_0 b}$$

$$\sigma^+ = \frac{\sigma}{\sigma_0}, \quad \lambda^+ = \frac{\lambda}{\lambda_0}, \quad \nu^+ = \frac{\nu}{\nu_0}$$

$$F_- = \frac{3\sigma_0}{2 j_0^2} k\delta n T \nu$$

Using these definitions we reduce the problem (for $I / kT_0 \gg 1$, $\partial / \partial z = 0$) to the following one:

$$\frac{\partial}{\partial x^+}\left[\frac{1}{\sigma^+}\frac{\partial\Phi^+}{\partial x^+} + \frac{\Omega}{\sigma^+}\frac{\partial\Phi^+}{\partial y^+}\right] + \frac{\partial}{\partial y^+}\left[\frac{1}{\sigma^+}\frac{\partial\Phi}{\partial y^+} - \frac{\Omega}{\sigma^+}\frac{\partial\Phi^+}{\partial x^+}\right] + \tag{1.2}$$

$$\frac{\partial}{\partial y^+}\frac{\Omega}{\sigma^+} - \frac{\partial}{\partial x^+}\frac{1}{\sigma^+} = 0$$

$$\frac{\partial\theta}{\partial t^+} - U^+ \frac{\partial\theta}{\partial y^+} - \nabla^+ \frac{\lambda^+}{1 + \Omega^2} \nabla^+ T^+ = \frac{1}{\sigma^+} - F_- +$$

$$\frac{1}{\sigma^+}\left[\left(\frac{\partial\Phi^+}{\partial x^+}\right)^2 + \left(\frac{\partial\Phi^+}{\partial y^+}\right)^2 - 2\frac{\partial\Phi^+}{\partial y^+}\right]$$

$$\frac{\partial\Phi^+(0, y, t)}{\partial y^+} = \frac{\partial\Phi^+(1, y, t)}{\partial y^+} = 0, \quad \theta(0, y, t) = \theta \quad (1, y, t) = 0 \tag{1.3}$$

We introduce the formulas for the coefficients of electric and thermal conductivities, drift velocity of electrons and collision frequency

$$(\sigma^+)^{-1} = 1 + a_{11}\theta + a_{12}\theta^2 + a_{13}\theta^3 + O(\theta^4) \tag{1.4}$$

$$\frac{\Omega}{\sigma^+} = \Omega_0 (1 + a_{21}\theta + a_{22}\theta^2 + a_{23}\theta^3 + O(\theta^4))$$

$$U^+ = U_0 + a_{31}\theta + a_{32}\theta^2 + O(\theta^3)$$

$$\frac{\lambda^+ n_0}{T_0 (1 + \Omega_0^2(\theta))}\frac{\partial T}{\partial n} = \frac{n_0}{T_0 (1 + \Omega_0^2)}\frac{\partial T_0}{\partial n_0} + a_{41}\theta + O(\theta^2)$$

The coefficients a_{ij} $(i, j = 1, 2, \ldots)$ of the expansion (1.4) are determined by the dependence of the collision cross sections on the electron temperature. Hereafter only dimensionless parameters will be used and therefore the superscript plus is omitted everywhere.

2. If the system of equations (1.2) is linearized at the homogeneous and stationary state (having first eliminated T^{+})

$$\theta(x, y, t) = 1 + \theta_0(x, y, t), \quad \Phi = \Phi_0(x, y, t), \quad |\Phi_0| < 1$$

then the functions θ_0, Φ_α will represent the solution of the following problem:

$$L_{11}\Phi_\alpha + L_{12}\theta_\alpha = 0, \quad L_{21}\Phi_0 + L_{22}\theta_\alpha = 0 \tag{2.1}$$
$$x \in [0, 1], \quad y \in (-\infty, \infty), \quad t \geqslant 0$$

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}, \quad L_{12} = \Omega_0 a_{21}\frac{\partial}{\partial y} - a_{11}\frac{\partial}{\partial x} \tag{2.2}$$
$$L_{21} = 2\frac{\partial}{\partial x}, \quad L_{22} = -\Lambda L_{11} + \frac{\partial}{\partial t} - U_0\frac{\partial}{\partial y} - f'$$
$$\Lambda = \frac{\lambda_0 n_0 \sigma_0}{j_0^2 b^2 (1+\Omega_0^2)}\frac{\partial T_0}{\partial n_0}, \quad f' = a_{11} - \frac{\partial F_-}{\partial \theta}$$

$$\frac{\partial \Phi_0(0, y, t)}{\partial y} = \frac{\partial \Phi_0(1, y, t)}{\partial y} = 0, \quad \theta_0(0, y, t) = \theta_0 \quad (1, y, t) = 0 \tag{2.3}$$

We shall find the solution of Eqs. (2.2) in the form

$$\Phi_0 = \varphi(x_0)\exp(iKy + pt), \quad \theta_0 = \theta(x)\exp(iKy + pt) \tag{2.4}$$

The characteristic equation for the eigenvalues of the problem has the form

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} & e^{\lambda_4} \\ \varkappa_1 & \varkappa_2 & \varkappa_3 & \varkappa_4 \\ \varkappa_1 e^{\lambda_1} & \varkappa_2 e^{\lambda_2} & \varkappa_3 e^{\lambda_3} & \varkappa_4 e^{\lambda_4} \end{vmatrix} = 0 \tag{2.5}$$

Here

$$\varkappa_j(\lambda_j) = \frac{i2\lambda_j}{f' + \Lambda(\lambda_j^2 - K^2)}, \quad j = 1, 2, 3, 4$$

where λ_j are the roots of the following equation:

$$\Lambda\lambda^4 - (p - 2\Lambda K^2 - f' + 2a_{11})\lambda^2 + i2\Omega_0 a_{21}\lambda + K^2(p - f' + \Lambda K^2) = 0 \tag{2.6}$$

The presence of the small parameter Λ at a higher derivative in (2.6) leads to a type of solution for a boundary layer near the channel wall; in this case the solution in the channel kernel depends weakly on the form of the temperature boundary conditions. Due to this fact and using Eqs. (2.5), (2.6), it is possible to show that p_n can be determined from the following approximate expression:

$$K^2(p_n + \Lambda K^2 - f')(p_n + 2\Lambda K^2 - f' + 2a_{11}) + (\Pi n)^2(p_n + 2\Lambda K^2 - f' + 2a_{11})^2 - (\Omega_0 K a_{11})^2 = 0 \tag{2.7}$$
$$n = 1, 2, \ldots$$

It follows from (2.7) that for the Hall parameters exceeding the critical value

$$\Omega_+ = \frac{K_n(2\Lambda K^2 - f' + 2a_{11})}{K\,|\,a_{21}\,|}$$

we have $\mathrm{Re}\,p > 0$ and the initial state is unstable.

The neutral curve ($\mathrm{Re}\,p = 0$) for an argon plasma ($T_0 = 4\ 000°\ K$, $P = 1$ atm) with an addition of cesium is shown in Fig. 1 we neglect the influence of the magnetic field on the coefficient of thermal conductivity $\Lambda^+ = \Lambda\,(1 + \Omega_+^2)$, $l_x = 20$ cm, $n = 1$) (*). The minimum critical value of the Hall parameter (corresponding to the extreme left position in Fig. 1) determines the beginning of the rise of instability. It weakly depends on the parameter Λ and its value is determined mainly by the values a_{11}, a_{12}, f'. The instability region H is limited by large values of the wave number K (small wavelengths) as well as by small values of K (large wavelengths). The instability region contracts with increasing thermal conductivity (the magnetic field induction is constant). The value of the wave number corresponding to the starting point of the rise of instability (point at $\Omega_{+\min}$) decreases with increasing parameter Λ. The lower and upper branches of the neutral curve for large values of the Hall parameter correspond to

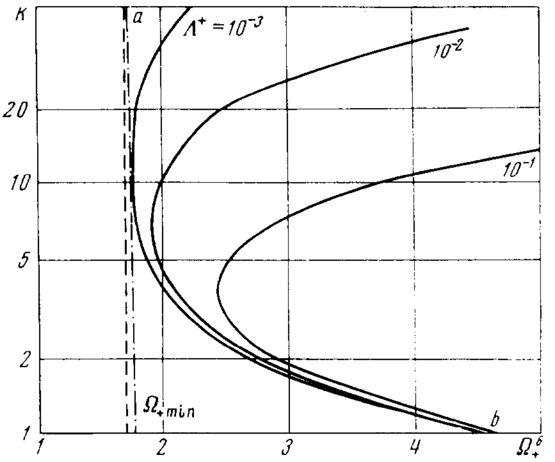

Fig. 1

$$K \sim 1\,/\,\Omega_+, \qquad K \sim \Omega_+^{1/4}$$

The stability for large wave numbers K can be explained by the stabilizing action of the thermal conductivity. The stability at small values of K is related to the fact that in this case the influence of the Hall effect vanishes (the lines of electric current become parallel). If the thermal conductivity is not taken into account, all the area on the right of the curve a, b (Fig. 1) belongs to the instability region.

3. To solve the question of the mode (magnetically hard or soft) of the loss of stability, it is necessary to take into account the nonlinear effects, i.e. to utilize the complete system of equations (1.2). We shall find the solution of system (1.2) for a self-oscillation state with boundary conditions (1.3) in the form

$$\Phi\,(x,\ Y), \quad \theta\,(x,\ Y), \qquad Y = y + Wt$$

expanding the functions in series with respect to the small parameter of the supercritical state

*) Computation of the neutral curve was performed by Iu. V. Trofimov.

$$\theta = 1 + \sum_{n=0} \varepsilon^{n+1}\theta_n, \qquad \Phi = \sum_{n=0} \varepsilon^{n+1}\Phi_n \tag{3.1}$$
$$W = U_0 + \sum_{n=1} \varepsilon^{n+1} W_n, \quad \Omega = \Omega_+ + \varepsilon^2$$

Instead of an expansion with respect to the small parameter of the supercritical state, a series expansion with respect to the small amplitude can be used, as it was done in [9]. These two expansions give the same results.

Substituting expansion (3.1) in Eq. (1.2) and equating the coefficients of the same degree in ε, we obtain the following system of equations for the functions θ_n and Φ_n:

$$L_{11}\Phi_n + L_{12}\theta_n = f_{1n}, \quad L_{21}\Phi_n + L_{22}{}^{+}\theta_n = f_{2n} \tag{3.2}$$
$$L_{22}{}^{+} = -\Lambda L_{11} - f'$$

The operators L_{11}, L_{12}, L_{21} are determined by the expressions (2.2) while for the functions f_{nm} the following formulas are valid

$$f_{10} = f_{20} = 0 \tag{3.3}$$
$$f_{11}(\Phi_0{}^2,\ \theta_0{}^2,\ \theta_0\Phi_0) = -\{a_{11}[\theta_{0x}\Phi_{0x} + \theta_{0y}\Phi_{0y} + \theta_0(\Phi_{0xx} + \Phi_{0yy})] + \Omega_+ a_{21}(\theta_{0x}\Phi_{0y} - \theta_{0y}\Phi_{0x}) + 2(a_{22}\Omega_+\theta_{0y} - a_{12}\theta_{0x})\theta_0\}$$
$$f_{21}(\Phi_0{}^2,\ \theta_0{}^2,\ \theta_0\Phi_0) = {}^1/_2 f''\theta_0{}^2 + \Phi_{0x}{}^2 + \Phi_{0y}{}^2 + W_1\theta_{0y} + \Phi_{0y}\theta_{0x} - 2a_{11}\theta_0\Phi_{0x} - a_{31}\theta_0\theta_{0y} + a_{41}(\theta_{0x}{}^2 + \theta_{0y}{}^2 + \theta_0\Delta\theta_0) - \theta_{0y}\Phi_{0x}$$
$$f_{12} = f_{11}(\Phi_1\Phi_0,\ \theta_1\theta_0,\ \Phi_1\theta_0,\ \Phi_0\theta_1) + 2a_{12}(\Phi_{0x}\theta_{0x} + \Phi_{0y}\theta_{0y}) + 2a_{22}\theta_0(\theta_{0x}\Phi_{0y} - \theta_{0y}\Phi_{0x})\Omega_+ + 3\Omega_+ a_{23}\theta_0{}^2\theta_{0x} - 3a_{13}\theta_0{}^2\theta_{0y} - a_{12}\theta_0\Delta\Phi_0 + a_{21}\theta_{0y}$$
$$f_{22} = f_{21}(\Phi_1\Phi_0,\ \theta_1\theta_0,\ \theta_1\Phi_0,\ \theta_0\Phi_1) - {}^1/_6 f'''\theta_0{}^3 - 2a_{12}\theta_0{}^2\Phi_{0x} + a_{11}\theta_0(\Phi_{0x}{}^2 + \Phi_{0y}{}^2) + a_{32}\theta_0{}^2\theta_{0y} + a_{42}[2\theta_0(\theta_{0x}{}^2 + \theta_{0y}{}^2) + \theta_0{}^2\Delta\theta_0] + \theta_0(\Phi_{0x}\theta_{0y} - \Phi_{0y}\theta_{0x}) - \frac{2\Omega_+}{1\Omega + \Omega_+{}^2}\Lambda\theta_0$$

The functions Φ_0, θ_0 satisfy the homogeneous system of equations and its solution can be represented in the form

$$\Phi_0(x,\ Y) = \alpha(\varphi(x)e^{iKY} + \varphi^*(x)e^{-iKY}) \tag{3.4}$$
$$\theta_0(x,\ y) = \alpha(\theta(x)e^{iKY} + \theta^*(x)e^{-iKY})$$

The functions $\varphi(x)$, $\theta(x)$ can be written in the form of asymptotic expansions with respect to the small parameter $\sqrt{\Lambda}$ (hereafter we shall use only the first terms of this expansion, although finding subsequent terms does not present any special difficulty). For the function φ there exists the following relation:

$$\varphi(x) = e^{icx}\sin\pi x + O(\sqrt{\Lambda})$$

To fulfil the boundary conditions for $\theta(x)$ it is necessary to take into account higher order derivatives and to construct the type of solution for a boundary layer at the wall. The general solution has the form of a multiple expansion

$$\theta(x) = \theta^{(1)}(x, \sqrt{\Lambda}) + O(\sqrt{\Lambda})$$

$$\theta^{(1)}_{(x)} = \frac{2}{f' - \Lambda K^2}\Big\{(\exp icx)(\pi\cos\pi x + ic\sin\pi x) -$$

$$\Big[\Big(\exp - \frac{\lambda_1 x}{\sqrt{\Lambda}}\Big) + \exp\Big(ic + \frac{x-1}{\sqrt{\Lambda}}\lambda_1\Big)\Big]\Big\}$$

Here

$$c = -\frac{\Omega_+ K a_{21}}{f' - 2a_{11} - 2\Lambda K^2}, \quad K_1 = (\pi^2 + K^2)^{1/2}$$

$$\lambda_1 = (\Omega_+ K \mid a_{11} \mid / K_1)^{1/2}$$

We note that in spite of the fact that the parameter Λ is much smaller than unity, the product ΛK can be considerable because of a chosen K.

In (3.4) the value of α is a constant (wave amplitude) which is to be determined. Substituting (3.4) into the relation for f_{11} and f_{21}, we obtain a linear homogeneous system of differential equations for the functions Φ_1 and θ_1. The condition for the solution of a nonhomogeneous system can be written in the form

$$\int_0^1 \int_0^{2\pi/K} (f_i q^*) \exp(-iKY)\, dx\, dY = 0 \tag{3.5}$$

$$f_i(f_{1i}, f_{2i}), \quad i = 1, 2, \ldots, \quad q^*(\varphi_0^+(x), \theta_0^+(x))$$

Here $\varphi_0^+(x)$, $\theta_0^+(x)$ is the solution of the conjugate system $(L_{22}^+ = L_{22}^*)$

$$L_{11}^*\varphi_0^+(x) + L_{12}^*\theta_0^+(x) = 0, \quad L_{21}^*\varphi_0^+ + L_{22}^*\theta_0^+ = 0$$

$$L_{21}^* = -2\frac{d}{dx}, \quad L_{12}^* = iK\Omega_+ a_{21} + a_{11}\frac{d}{dx}, \quad L_{11}^* = \frac{d}{dx^2} - K^2$$

with the boundary conditions $\varphi_0^+(0) = \varphi_0^+(1) = 0$, $\theta_0^+(0) = \theta_0^+(1) = 0$. From the condition for the system solution for $n = 1$, it follows that $W_1 = 0$, and for the functions Φ_1, θ_1 the formulas

$$\Phi_1(x, Y) = \alpha^2(\varphi_1(x)e^{2iKY} + \varphi_1^*(x)e^{-2iKY} + \varphi_1^-(x)) \tag{3.6}$$

$$\theta_1(x, Y) = \alpha^2(\theta_1(x)e^{2iKY} + \theta_1^*(x)e^{-2iKY} + \theta_1^-(x))$$

are valid. Here φ_1, θ_1, φ_1^-, θ_1^- satisfy the following equations:

$$L_{11}^*\varphi_1 + L_{12}'\theta_1 = f_{11}(\varphi^2, \theta^2, \varphi\theta), L_{21}\varphi_1 + L_{22}^*\theta_1 = f_{21}(\varphi^2, \theta^2, \varphi\theta)$$

$$G_{11}\varphi_1^- + G_{12}Q_1^- = f_{11}(\varphi\varphi^*, \theta\theta^*, \varphi\theta^*, \varphi^*\theta)$$

$$L_{21}\varphi_1^- + G_{22}\theta_1^- = f_{21}(\varphi\varphi^*, \theta\theta^*, \varphi\theta^*, \varphi^*\theta)$$

$$L_{12}' = i\Omega_+ a_{21}K - a_{11}\frac{d}{dx}, \quad G_{11} = \frac{d^2}{dx^2}$$

$$G_{12} = -a_{11}\frac{d}{dx}, \quad G_{22} = -\Lambda G_{11} - f'$$

Substituting (3.1) and (3.6) into the expansions for f_{12}, f_{22}, we obtain the system of equations for the functions Φ_2 and θ_2. The condition for the solution of (3.5) for $n = 2$ permits the amplitude α and W to be determined. Taking only the main contributing terms, we can write

$$\alpha^2 = \frac{16\Lambda K^2\chi}{a_{11}\Omega_+\chi_1} \tag{3.7}$$

$$W_2 = [(\Omega_+K - c)(2\Lambda K^2 - f') - 2\Lambda K\pi^2](2\chi_1)^{-1}$$

$$\chi_1 = (\Omega_+K - c)^2 - \pi^2, \quad \chi = [(2\Lambda K_1{}^2 - f') - 2K_1\Lambda(\Omega_+K + K_1)](\Lambda K^2 - f')^{-1}$$

Here χ_1 is a positive function of a constant sign for different values of the wave number K and of the Hall parameter. The function χ is positive for large wave numbers and negative for

$$K < K_+ = \left(\frac{-f'}{2\Lambda\Omega_+}\right)^{1/2}$$

The negative sign of the function χ indicates that in this case a hard mode of the loss of stability is present and a self-oscillatory mode exists for $\Omega_0 < \Omega_+$. In this case it is necessary to use the expansion $\Omega_0 = \Omega_+ - \varepsilon^2$ and the right side of the formula (3.7) will be preceded by the plus sign.

The minimal critical value of the Hall parameter Ω_{+min} takes place in the region of a magnetically soft mode of the loss of stability, which was proved by the experiments [10, 11]. With the Hall parameter further increasing a magnetically hard mode of the loss of stability can occur as the neutral curve is crossed.

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REFERENCES

1. Velikchov, E.P. and Dykhne, A.M., Plasma turbulence due to ionization instability in a strong magnetic field. Compte Rendu 6-e Conférence Internat. Phénomènes ionisation gaz, Vol.4, Paris, 1963.

2. Kerrebrock, J.L., Nonequilibrium ionization due to electron heating. 1.Theory. AIAA Journal, Vol.2, №6, 1964.

3. Shipuk, I.Ia., Instability ionization in a disk channel of a nonequilibrium MHD generator. Collection: Magnitogidrodinamicheskii Metod Polucheniia Electroenergii. Moscow, "Energiia", 1968.

4. Dykhne, A.M., Ionization instability in a bounded region. Electr. MHD, Vol.1, Vienna, 1968.

5. Shioda, S. and Heruya, I., Electrothermal instability with effects of electron thermal conduction and wall boundaries. 12th Nat.Symposium Engng Aspects MHD, Aragonne Illinois, USA, 1972.

6. Velikhov, E.P., Degtiarev, L.M., Samarskii, A.A. and Favorskii, A.P., Calculation of ionization instability in a low temperature magnetized plasma. Dokl. Akad. Nauk SSSR, Vol.184, №3, 1969.

7. Lengyel, L.L., On the numerical simulation of ionization instability. Physics Letters, Vol.29A, №2, p.60, 1969.

8. Nedospasov, A.V., Striae. Uspekhi fiz.nauk, Vol.94, №3, 1968.

9. Landau, L.D., On turbulence problem. Dokl. Akad. Nauk SSSR, Vol.44, №8, 1944.

10. Shipuk, I.Ia. and Pashkin, S.V., Ionization instability of a plasma in crossed fields. Dokl. Akad. Nauk SSSR, Vol.176, №6, 1967.

11. Riedmüller, W., Experimental investigation of instabilities in a potassium-seeded argon plasma in crossed electric and magnetic fields. In: Electr. MHD, Vol.1, Vienna, 1968.

Translated by H.B.